ECON220B Discussion Section 1

Basics of Asymptotic Distribution

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Intro

- I am a second year interested in Macro and Time Series Econometrics.
- We will meet every Thursday 5/6 pm. The room is booked until 7pm (office hour).
- If you have doubts or need help, I am always available at lbini@ucsd.edu
- ECON220B is challenging: 6 problem sets, midterm + final exam (a lot of material).
- My goal: cover crucial topics for final exam/qual plus tools that might be helpful for your future research.

Roadmap

1. Convergence theorems.
2. Tools to study asymptotic distribution of estimators.
3. Exercise Asymptotic Linear Representation.
4. Superefficient Estimator.

Convergence Theorems

1. WLLN: If $\{x_1, \ldots, x_n\}$ iid where $E[x_i] = \mu < \infty$, then $\forall \varepsilon > 0$ $\lim_{n \to \infty} \mathcal{P}(|\bar{x}_n - \mu| \geq \varepsilon) = 0$, i.e. $\bar{x}_n \xrightarrow{p} \mu$.

2. Small Oh-Pee: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of RVs, If $x_n \xrightarrow{p} 0$ then we say $x_n = o_p(1)$.

3. CLT: If $\{x_1, \ldots, x_n\}$ iid, $x_i \sim (\mu, \sigma^2)$, where $\mu < \infty$, $\sigma^2 < \infty$, then $\sqrt{n}(\bar{x} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$.

4. Big Oh-Pee: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of RVs, If $x_n \xrightarrow{d} \mathcal{L}$ then we say $x_n = O_p(1)$.

Tools Asymptotic Distribution (1/3)

- Slutsky's theorem: let X_n,Y_n be a sequence of scalar/vector/matrix random elements, If $X_n \xrightarrow{p} x$ and $Y_n \xrightarrow{d} Y$ then:

- Relationship between $o_p(1)$ and $O_p(1)$:

- Continuous mapping theorem let X_n be a sequence of scalar/vector/matrix random elements and $g(\cdot)$ be a continuous function, If $X_n \xrightarrow{p} x$ then $g(X_n) \xrightarrow{p} g(x)$

Tools Asymptotic Distribution (2/3)

- Taylor's Mean Value Theorem: Let $g : \mathbb{R}^m \to \mathbb{R}$, if g continuous in $[\theta, \hat{\theta}]$, differentiable in $(\theta, \hat{\theta})$, then:

$$g(\hat{\theta}) = g(\theta) + \nabla g(\tilde{\theta})'\big(\hat{\theta} - \theta\big) \quad \text{with:} \quad \tilde{\theta} \in [\theta, \hat{\theta}]$$

- Delta Method: just trivial manipulation:

$$\begin{aligned}
\sqrt{n}\big(\hat{\theta} - \theta\big) &\xrightarrow{d} \mathcal{N}(0, \Sigma) \\
\sqrt{n}\big(g(\hat{\theta}) - g(\theta)\big) &= \nabla g(\tilde{\theta})'\sqrt{n}\big(\hat{\theta} - \theta\big) \\
\sqrt{n}\big(g(\hat{\theta}) - g(\theta)\big) &\xrightarrow{d} \mathcal{N}(0, \nabla g(\theta)'\, \Sigma\, \nabla g(\theta))
\end{aligned}$$

Tools Asymptotic Distribution (3/3)

- Asymptotic linear representation: fantastic tool to study asymptotic distribution of estimators. Based on the so-called "Levy-Lindeberg" CLT:

$$\sqrt{n}(\bar{x}-\mu) = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{n}{n}\mu\right] = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(x_i-\mu) \xrightarrow{d} \mathcal{N}(0,\sigma^2)$$

- Why is it useful? Suppose $\hat{\theta} = \left(\sum_{i=1}^{n} x_i\right)/\left(\sum_{i=1}^{n} y_i\right)$, can we apply CLT?
- We want to derive the influence function $\psi(x_i, y_i)$ such that:

$$\sqrt{n}(\hat{\theta}-\theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi(x_i,y_i) + o_p(1)$$

Exercise on Asymptotic Linear approximation

- Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be independently and identically distributed, where both x_i and y_i are univariate but they may not be independent. We define the following parameter of interest and the estimators:

$$\theta = \frac{E[x_i]}{E[y_i]} = \frac{\mu}{\nu}, \quad \hat{\theta} = \frac{\bar{x}}{\bar{y}}, \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

(1) Derive the asymptotic linear representation of $\hat{\theta}$.

(2) Derive the asymptotic distribution of $\hat{\theta}$.

(3) Propose an estimator for the asymptotic variance.

(4) Use the delta method to derive the asymptotic distribution, and compare with your previous answers.

Hodge's Estimator

- Assume we have an iid sample, $\{x_1, x_2, \dots, x_n\}$, from a distribution with mean μ and variance σ^2. To estimate μ, we will consider two estimators. The first one is just the sample mean $\hat{\mu} \equiv \bar{x}$, while the second one is called superefficient estimator:

$$\tilde{\mu} = \begin{cases} \hat{\mu} & \text{if } |\hat{\mu}| > n^{-1/4} \\ 0 & \text{if } |\hat{\mu}| \le n^{-1/4} \end{cases}$$

(1) Derive the asymptotic distribution of $\hat{\mu}$ and the asymptotic MSE.

(2) Show that $\mu \neq 0 \Rightarrow \mathbb{P}(\tilde{\mu} = \hat{\mu}) \to 1$ and $\therefore \sqrt{n}(\tilde{\mu} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$

(3) Show that $\mu = 0 \Rightarrow \mathbb{P}(\tilde{\mu} = \hat{\mu}) \to 0$ and $\therefore \sqrt{n}(\tilde{\mu} - \mu) \xrightarrow{p} 0$.

(4) Assume $\mu = cn^{-1/3}$, what is the asymptotic MSE of $\tilde{\mu}$?

Deceiving Asymptotics

Let's assume $\{x_1, x_2, \ldots, x_n\}$ iid with $x_i \sim \mathcal{N}(\mu, 1)$. This is the asymptotic MSE (scaled by the rate n) of the Hodge's estimator for different values of μ.