Practice Test

Moving Average of Order 1

Let $\{\varepsilon_t\}_{t=1}^{\infty}$ be an independent and identically distributed sequence of white noise, i.e. $\varepsilon_t \sim iid\ \mathcal{N}(0, \sigma^2)$, consider the following discrete-time stochastic process:

$$y_t = c + \varepsilon_t + \theta\varepsilon_{t-1} \tag{1}$$

also known as moving average of order 1, MA(1), with deterministic intercept c.

Question 1

Suppose we are interested in the sample average $(\hat{\mu})$ of the observed outcomes (y_t) for $t = 1, \ldots, T$. Can you apply WLLN to study the probability limit of the estimator? Find the probability limit of the estimator $\hat{\mu}$.

Question 2

Can you apply CLT? Under what assumption would you be able to use it?

Question 3

Informally, the CLT can be applied to dependent data if the degree of dependency does not grow with t or grows slowly enough as t increases. However, to study the asymptotic distribution of our estimator, we need to take a step back and study the properties of the stochastic process y_t.

Find the unconditional distribution of y_t, along with its mean and variance. Think carefully: do you need the CLT? Hint: ε_t and ε_{t-1} are independent.

Question 4

Find the conditional distribution of y_t given the realization from the previous period y_{t-1}. What does this distribution imply about dependence?

Question 5

Suppose we now have a Vector Moving Average of order 1 defined as

$$Y_t = C + \epsilon_t + \Theta\epsilon_{t-1} \tag{2}$$

$$Y_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} \quad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \quad \epsilon_t = \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right)$$

Find the distribution of Y_t.

Question 6

Now suppose that we have $Y_t = |y_t \quad y_{t-1}|'$. Find the unconditional distribution of Y_t. Hint: combine the results from question 3 and 5 - you only need to estimate one additional parameter.

Question 7

Let's extend the problem further: suppose you now define $Y_t = |y_t \;\; y_{t-1} \quad \cdots \quad y_{t-h}|'$ for a very large h. Find the unconditional distribution of Y_t. Note that now the dimensions of C, ϵ_t, Θ now depend on the number of lags considered, h.

Question 8

We are almost ready to apply the CLT to our estimator $\hat{\mu}$. We need to verify two conditions:

1. $E[y_t] = k$ where k is a constant, $\forall t$.
2. $\lim_{t\to\infty} \; t\, Var(\hat{\mu}) < \infty$

The second condition holds if the stochastic process satisfies the so-called absolute summability property. Given our stochastic process y_t, define the autocovariance function as:

$$\gamma(h) = Cov(y_t, y_{t-h})$$

where h is the lag time. The stochastic process is absolutely summable if $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$. Verify whether the MA(1) process satisfies the abovementioned conditions.

Question 9

Given $a_t(\hat{\mu} - p)$ where a_t is the appropriate convergence rate and p is the probability limit of $\hat{\mu}$, apply the CLT and compute the variance. Hint: Substituting the MA inside the summation and expanding the sum will cause a pattern to appear.

Question 10

Claim: y_t can be represented as an autoregressive process of order infinity:

$$y_t = \mu + \varepsilon_t + \sum_{i=1}^{\infty} \psi_i \, y_{t-i}$$

Derive the AR(∞) representation of y_t.

How could the covariance $Cov(y_t, y_{t-k})$ be equal to zero? We know that y_{t-k} appears in the infinite sum. Is this a contradiction?

$$y_t = c + \varepsilon_t + \theta\,\varepsilon_{t-1}$$

$$y_{t-1} = c + \varepsilon_{t-1} + \theta\,\varepsilon_{t-2}$$

① $\hat{\mu} \xrightarrow{P} ?$ $\qquad \hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} y_t$

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} c + \varepsilon_t + \theta\,\varepsilon_{t-1} \xrightarrow{P} c + \underbrace{E[\varepsilon_t]}_{=0} + \underbrace{E[\theta_1 \varepsilon_{t-1}]}_{\theta_1 E[\varepsilon_{t-1}],\ =0} = c$$

② $$\sqrt{T}(\hat{\mu} - c) = \sqrt{T}\left[\left(\frac{1}{T}\sum y_t\right) - \frac{Tc}{T}\right]$$

$$= \frac{1}{\sqrt{T}}\sum(y_t - c) = \frac{1}{\sqrt{T}}\sum \varepsilon_t + \theta\,\varepsilon_{t-1}$$

③ $N(0, \sigma^2(1+\theta^2))$?

$$y_t = c + \bar{\varepsilon}_t + \theta_1 \bar{\varepsilon}_{t-1}$$

$$M_{X+Y}(t) = E[e^{t(X+Y)}]$$

$$= E[e^{tX}]\,E[e^{tY}]$$

$$= e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}$$

$$E[y_t] = E[c + \varepsilon_t + \varepsilon_{t-1}\theta] = c$$

$$\mathrm{Var}(y_t) = \mathrm{Var}(\varepsilon_t) + \theta_1^2\,\mathrm{Var}(\varepsilon_{t-1})$$

$$= \sigma^2(1+\theta^2)$$

$$= e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$$

$$\mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

4

$$y_t \mid y_{t-1} = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} \mid y_{t-1}$$

$$= \underbrace{c + \theta_1 \varepsilon_{t-1}}_{K} + \varepsilon_t \mid y_{t-1}$$

ε_{t-1}

ε_{t-2}

$\{\varepsilon_t\}$ (IID)

$$= K + \varepsilon_t$$

$$K + N(0, \sigma^2)$$

$$y_t \mid y_{t-1} \sim N(c + \theta_1 \varepsilon_{t-1} ; \sigma^2)$$

5

$$Y_t = c + \underline{\epsilon_t} + \Theta\, \underline{\epsilon_{t-1}} \qquad \epsilon_t \sim N\left(\cdot, \begin{vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{vmatrix}\right)$$

$$E[Y_t] = c \qquad \{\epsilon_t\}_{t=1}^{T} \text{ IID}$$

$$\underbrace{Var(Y_t)} = \underbrace{Var(\epsilon_t)}_{\Sigma} + \Theta \Sigma \Theta^T$$

$\uparrow$ diagonal

$$\begin{vmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{vmatrix} \begin{vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{vmatrix} \begin{vmatrix} \theta_{11} & \theta_{21} \\ \theta_{12} & \theta_{22} \end{vmatrix}$$

NOT DIAGONAL

$$\Theta \quad \underbrace{\begin{vmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{vmatrix}}_{\text{PROPAGATION}} \begin{vmatrix} MP_t \\ FIS_t \end{vmatrix}$$

6

$$Y_t = c + \epsilon_t + \Theta\, \epsilon_{t-1}$$

$$\begin{vmatrix} y_t \\ y_{t-1} \end{vmatrix} = \begin{vmatrix} c \\ c \end{vmatrix} + \begin{vmatrix} \epsilon_t \\ \epsilon_{t-1} \end{vmatrix} + \begin{vmatrix} \theta_1 & 0 \\ 0 & \theta_1 \end{vmatrix} \begin{vmatrix} \epsilon_{t-1} \\ \epsilon_{t-2} \end{vmatrix}$$

$$\begin{vmatrix} y_t \\ y_{t-1} \end{vmatrix} = \begin{vmatrix} c \\ c \end{vmatrix} + \underbrace{\begin{vmatrix} 1 & \theta & 0 \\ 0 & 1 & \theta \end{vmatrix}}_{A} \underbrace{\begin{vmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \end{vmatrix}}_{u_t} \quad *$$

$$Y_t \sim N\left(\begin{vmatrix} c \\ c \end{vmatrix}, V \right)$$

$$N\left(\begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}, \underbrace{\sigma_2 I_3}_{V_u} \right)$$

$$V = A V_u A^T$$

$$Var(y_t) = (1+\theta^2)\sigma^2$$

$\{\varepsilon_t\}$ IID

$$Var(y_{t-1}) = (1+\theta^2)\sigma^2$$

$$Cov(y_t, y_{t-1}) = Cov(c + \varepsilon_t + \theta\varepsilon_{t-1}, c + \varepsilon_{t-1} + \theta\varepsilon_{t-2})$$

$$= Cov(\theta\varepsilon_{t-1}, \varepsilon_{t-1}) = \theta Var(\varepsilon_{t-1}) = \theta\sigma^2$$

$$Y_t \sim N\left(\begin{vmatrix} c \\ c \end{vmatrix}, \begin{vmatrix} (1+\theta^2)\sigma^2 & \theta\sigma^2 \\ \theta\sigma^2 & (1+\theta^2)\sigma^2 \end{vmatrix} \right)$$

$$y_t = c + \varepsilon_t + \theta\,\varepsilon_{t-1}$$

$$y_{t-1} = c + \varepsilon_{t-1} + \theta_1\,\varepsilon_{t-2}$$

$$\begin{array}{c} y_t \\ y_{t-1} \end{array} \begin{vmatrix} 1 & \theta & 0 \\ 0 & 1 & \theta_1 \end{vmatrix} \begin{vmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \end{vmatrix}$$

7

$$\begin{vmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-n} \end{vmatrix} = \underset{= C}{\begin{vmatrix} c \\ c \\ \vdots \\ c \end{vmatrix}} \overbrace{\underset{n \times (n+1)}{\begin{vmatrix} 1 & \theta & 0 & \cdots & & 0 \\ 0 & 1 & \theta & 0 & \cdots & 0 \\ 0 & 0 & 1 & \theta & 0 \cdots & 0 \\ & & & & & \end{vmatrix}}}^{:= A} \underset{(n+1)\times 1}{\begin{vmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \vdots \\ \varepsilon_{t-n-1} \end{vmatrix}}$$

$$\mathrm{Cov}(c + \varepsilon_t + \theta \varepsilon_{t-1},\ c + \varepsilon_{t-2} + \theta \varepsilon_{t-3})$$

$$N\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \sigma^2 I_{n+1} \right)$$

$$E[Y_t] = c$$

$$\mathrm{Var}(Y_t) = A V_\varepsilon A^T \qquad (y_{t-2})$$

$$\begin{array}{r} y_t \\ y_{t-1} \\ (y_{t-2}) \\ \\ \end{array} \begin{bmatrix} (1+\theta^2)\sigma^2 & \theta\sigma^2 & 0 & \\ \theta\sigma^2 & (1+\theta^2)\sigma^2 & \theta\sigma^2 & \\ 0 & \theta\sigma^2 & (1+\theta^2)\sigma^2 & \\ 0 & 0 & & \ddots \end{bmatrix}$$

$\mathrm{Cov}($

(8) $E[y_t] = K \quad \forall t$ (*)

$$\underbrace{\lim_{T \to \infty} T \cdot Var(\hat{\mu})} < \infty \quad *$$

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty \quad (*)$$

$\gamma(k) := Cov(y_t, y_{t-k})$

$$\underbrace{\sum_{k=-\infty}^{\infty} |\gamma(k)|} = |\gamma(\overset{*}{-1})| + |\gamma(0)| + |\gamma(\overset{*}{1})|$$

$$= 2|\theta\sigma^2| + (1+\theta^2)\sigma^2$$

$$= 2|\theta|\sigma^2 + (1+\theta^2)\sigma^2 < \infty$$

$\theta^2 < \infty$

9 $\sqrt{T}(\hat{\mu} - c) \overset{a}{\sim} N(0, V)$

$$= V = Var\left(\sqrt{T}\left(\hat{\mu} - c\right)\right) = T\, Var(\hat{\mu} - c) = T\, Var(\hat{\mu})$$

$$= T\, Var\left(\frac{1}{T}\sum y_t\right) = \frac{T}{T^2} Var\left(\sum y_t\right) = \frac{1}{T} Var\left(\sum_{t=1}^{T} y_t\right)$$

$$= \frac{1}{T}\left[T\, Var(y_t) + 2(T-1)\gamma(1) + 2(T-2)\gamma(2) \cdots\right.$$

$$= Var(y_t) + 2\frac{(T-1)}{T}\gamma(1) + \ldots + \frac{2(T-K)}{T}\gamma(K)$$

$$= \gamma(0) + 2\gamma(1) + \ldots + 2\gamma(K)$$

$$= \gamma(0) + \gamma(1) + \gamma(-1) \cdots$$

$$= \sum_{-\infty}^{\infty}\gamma(K) \rightarrow (1+\theta^2)\sigma^2 + 2\theta\sigma^2 = V$$

$$\sqrt{T}(\hat{\mu} - c) \xrightarrow{d} N\left(0, V = (1+\theta^2)\sigma^2 + 2\theta\sigma^2\right)$$

10

Note : $y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$

$$y_{t-1} = c + \varepsilon_{t-1} + \theta_1 \varepsilon_{t-2} \longrightarrow \varepsilon_{t-1} = y_{t-1} - c - \theta_1 \varepsilon_{t-2}$$

$$y_{t-2} = c + \varepsilon_{t-2} + \theta_1 \varepsilon_{t-3} \longrightarrow \varepsilon_{t-2} = y_{t-2} - c - \theta_1 \varepsilon_{t-3}$$

NOW WE RECURSIVELY SUBSTITUTE THE ERROR TERMS

$$\therefore \quad y_t = c + \varepsilon_t + \theta_1 \overbrace{(y_{t-1} - c - \theta_1 \varepsilon_{t-2})}^{:= \varepsilon_{t-1}} = \varepsilon_t + c - \theta_1 c + \theta_1 y_{t-1} - \theta_1^2 \varepsilon_{t-2}$$

$$y_t = \varepsilon_t + c - \theta_1 c + \theta_1 y_{t-1} - \theta_1^2 \overbrace{(y_{t-2} - c - \theta_1 \varepsilon_{t-3})}^{:= \varepsilon_{t-2}}$$

$$y_t = \varepsilon_t + c - \theta_1 c + \theta_1^2 c + \theta_1 y_{t-1} - \theta_1^2 y_{t-2} + \theta_1^3 \varepsilon_{t-3}$$

$$\vdots$$

$$y_t = \varepsilon_t + c \underbrace{\sum_{i=0}^{\infty} (-\theta_1)^i}_{\text{CONVERGENT GEOMETRIC SEQUENCE}} + \sum_{i=1}^{\infty} -(-\theta_1)^i y_{t-i} \qquad \text{IF } |\theta_1| < 1 \text{ THEN}$$

$$y_t = \varepsilon_t + \frac{1}{1-(-\theta_1)} c + \sum_{i=1}^{\infty} \psi_i y_{t-i} \qquad \text{WHERE } \psi_i := -(-\theta_1)^i$$

$$y_t = \varepsilon_t + \mu + \sum_{i=1}^{\infty} \psi_i y_{t-i} \qquad \text{WHERE } \mu := \frac{c}{1+\theta}$$

A MORE ELEGANT PROOF REQUIRES LAG OPERATOR That you don't Know, so I'll skip it.

LET'S SEE WHAT HAPPENS TO THE COVARIANCE when $K > 1$

$$Cov(y_t, y_{t-k}) = Cov\left(\varepsilon_t + \mu + \sum^{\infty} \psi_i y_{t-i}, y_{t-k}\right)$$

ONLY $\gamma(1)$ $\gamma(0)$ $\gamma(-1)$ SURVIVE !!

$$= Cov\left(\sum_{i=1}^{\infty} \psi_i y_{t-i}, y_{t-k}\right)$$

REMOVE CONSTANT AND ε_t CLEARLY UNCORRELATED WITH y_{t-k}

$$= \psi_{k-1} \underbrace{Cov(y_{t-k+1}, y_{t-k})}_{\gamma(1)} + \psi_k \underbrace{Cov(y_{t-k}, y_{t-k})}_{\gamma(0)} + \psi_{k+1} \underbrace{Cov(y_{t-k-1}, y_{t-k})}_{\gamma(1)}$$

$$= \psi_{k-1} \theta_1 \sigma^2 + \psi_k (1+\theta_1^2)\sigma^2 + \psi_{k+1} \theta_1^2 \sigma^2$$

$$= -(-\theta)^{k-1} \theta_1 \sigma^2 + -(-\theta)^k (1+\theta_1^2)\sigma^2 + -(-\theta)^{k+1} \theta_1 \sigma^2$$

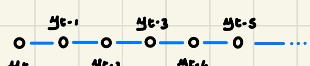
$$= -(-\theta)^k \left\{ -\frac{1}{\theta_1} \theta_1 \sigma^2 + (1+\theta_1^2)\sigma^2 - \theta_1 \theta_1 \sigma^2 \right\}$$

$$= \frac{-(-\theta_1)^k}{\theta_1} \left\{ -\theta_1 \sigma^2 + \theta_1 (1+\theta_1^2)\sigma^2 - \theta_1^3 \sigma^2 \right\}$$

$$= \frac{-(-\theta_1)^k}{\theta_1} \left\{ \overline{-\theta_1 (1+\theta^2)\sigma^2} + \theta_1 (1+\theta_1)^2 \sigma^2 \right\} = 0$$

$\therefore$ ROBUST TO OUR PREVIOUS FINDINGS

AR(∞) SUMMARIZES THE WHOLE DEPENDENCE STRUCTURE OF our process :



BUT The covariances are different from zeros only within 1 TIME LAG