

ECON220B Discussion Section 1

Basics of Asymptotic Distribution

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INTRO

- ▶ We will meet every **Monday** from 11:00 AM to 12:30 PM. The room is booked until 1:00 PM (office hour).
- ▶ If you have doubts or need help, I am always available at lbini@ucsd.edu or stop by my office **E460**
- ▶ ECON220B is challenging: 6 problem sets, midterm + final exam (a lot of material).
- ▶ My goal: cover crucial topics for final exam/qual plus tools that might be helpful for exams and research.

ROADMAP

1. **Convergence theorems.**
2. Tools to study the **asymptotic distribution** of estimators.
3. Exercises on **asymptotic distribution.**
4. Bonus: Superefficient Estimator.

CONVERGENCE THEOREMS

- ▶ **WLLN**: If $\{x_i\}_{i=1}^n$ iid where $E[x_i] = \mu < \infty$, then $\forall \varepsilon > 0 \lim_{n \rightarrow \infty} \mathcal{P}(|\bar{x}_n - \mu| \geq \varepsilon) = 0$, also written as $\bar{x}_n \xrightarrow{p} \mu$.
- ▶ **Small Oh-Pee**: Let $\{x_n\}_{n=1}^\infty$ be a sequence of RVs, If $x_n \xrightarrow{p} 0$ then we say $x_n = o_p(1)$.
- ▶ **CLT**: If $\{x_i\}_{i=1}^n$ iid, $x_i \sim (\mu, \sigma^2)$, $\mu < \infty$, $\sigma^2 < \infty$, then $\sqrt{n}(\bar{x} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$.
- ▶ **Big Oh-Pee**: Let $\{x_n\}_{n=1}^\infty$ be a sequence of RVs, If $x_n \xrightarrow{d} \mathcal{L}$ then we say $x_n = O_p(1)$.

TOOLS ASYMPTOTIC DISTRIBUTION (1/3)

► Slutsky's theorem

Let X_n, Y_n be a sequence of scalar/vector/matrix random elements, If $X_n \xrightarrow{p} x$ and $Y_n \xrightarrow{d} Y$ then:

1. $Y_n X_n \xrightarrow{d} Y x$
2. $Y_n + X_n \xrightarrow{d} Y + x$
3. $Y_n / X_n \xrightarrow{d} Y / x$

► Continuous mapping theorem

Let X_n be a sequence of scalar/vector/matrix random elements and $g(\cdot)$ be a continuous function. If $X_n \xrightarrow{p} x$ then $g(X_n) \xrightarrow{p} g(x)$

► Taylor's Mean Value Theorem

Let $g : \mathbb{R}^m \rightarrow \mathbb{R}$, if g continuous in $[\alpha, \hat{\alpha}]$, differentiable in $(\alpha, \hat{\alpha})$, then:

$$g(\hat{\alpha}) = g(\alpha) + \nabla g(\tilde{\alpha})'(\hat{\alpha} - \alpha) \quad \text{with: } \tilde{\alpha} \in [\alpha, \hat{\alpha}]$$

► Delta Method (just Taylor + Slutsky)

$$\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

$$\sqrt{n}(g(\hat{\alpha}) - g(\alpha)) = \nabla g(\tilde{\alpha})' \sqrt{n}(\hat{\alpha} - \alpha)$$

$$\sqrt{n}(g(\hat{\alpha}) - g(\alpha)) \xrightarrow{d} \mathcal{N}(0, \nabla g(\alpha)' \Sigma \nabla g(\alpha))$$

TOOLS ASYMPTOTIC DISTRIBUTION (3/3)

► Asymptotic linear representation

Fantastic tool to study asymptotic distribution of estimators. Based on the so-called "Levy-Lindeberg" CLT:

$$\sqrt{n}(\bar{x} - \mu) = \sqrt{n} \left[\frac{1}{n} \sum_{i=1}^n x_i - \frac{n}{n} \mu \right] = \frac{1}{\sqrt{n}} \sum_{i=1}^n (x_i - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

- We want to derive the **influence function** $\psi(x_i, y_i)$ such that:

$$\sqrt{n}(\hat{\theta} - \theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi(x_i, y_i) + o_p(1)$$

- Are there any **other methods**?

Consider $\hat{\theta} = (\sum_{i=1}^n x_i) / (\sum_{i=1}^n y_i)$, do you know other tools to find **asymptotic distribution**?

EXERCISE ON ASYMPTOTIC DISTRIBUTION (1/2)

Let $\{(x_i, y_i)\}_{i=1}^n$ be a sequence of independently and identically distributed random variables, where both x_i and y_i are univariate but they may not be independent. We define the following objects of interest:

$$\theta = \frac{E[x_i]}{E[y_i]} = \frac{\mu}{\nu}, \quad \hat{\theta} = \frac{\bar{x}}{\bar{y}}, \quad \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i, \quad \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$

- (1) Derive the **asymptotic linear representation** of $\hat{\theta}$.
- (2) Derive the **asymptotic distribution** of $\hat{\theta}$.
- (3) Propose an estimator for the **asymptotic variance**.
- (4) Use the **delta method** to derive the asymptotic distribution, and compare with your previous answers.

EXERCISE ON ASYMPTOTIC DISTRIBUTION (2/2)

Let $\{x_i\}_{i=1}^n$ iid sample from a univariate distribution, $x_i \in \mathbb{R}$. Denote by $F(\cdot)$ the cumulative distribution which is defined as:

$$F(x) = \mathbb{P}(x_i \leq x)$$

For simplicity we will assume that x_i is continuously distributed on the unit interval such that:

- ▶ $F(x) = 0$ for all $x \leq 0$.
- ▶ $F(x) = 1$ for all $x \geq 1$.
- ▶ $F(x)$ continuous and strictly increasing for all $x \in (0, 1)$.

Define the **empirical CDF** as $\hat{F}(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{x_i \leq x\}$

- (1) Assume $0 < x < 1$, find the asymptotic distribution of $\hat{F}(x)$
- (2) Assume $0 < x \neq x' < 1$, find the joint asymptotic distribution of $\hat{F}(x)$ and $\hat{F}(x')$.

EXTRA: HODGE'S ESTIMATOR

- Assume we have an iid sample, $\{x_1, x_2, \dots, x_n\}$, from a distribution with mean μ and variance σ^2 . To estimate μ , we will consider two estimators. The first one is just the sample mean $\hat{\mu} \equiv \bar{x}$, while the second one is called **superefficient estimator**:

$$\tilde{\mu} = \begin{cases} \hat{\mu} & \text{if } |\hat{\mu}| > n^{-1/4} \\ 0 & \text{if } |\hat{\mu}| \leq n^{-1/4} \end{cases}$$

- (1) Derive the asymptotic distribution of $\hat{\mu}$ and the asymptotic MSE.
- (2) Show that $\mu \neq 0$ implies $\mathbb{P}(\tilde{\mu} = \hat{\mu}) \rightarrow 1$ and $\therefore \sqrt{n}(\tilde{\mu} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$
- (3) Show that $\mu = 0$ implies $\mathbb{P}(\tilde{\mu} = \hat{\mu}) \rightarrow 0$ and $\therefore \sqrt{n}(\tilde{\mu} - \mu) \xrightarrow{p} 0$.
- (4) Assume $\mu = cn^{-1/3}$, what is the asymptotic MSE of $\tilde{\mu}$?

EXTRA: DECEIVING ASYMPTOTICS

Let's assume $\{x_1, x_2, \dots, x_n\}$ iid with $x_i \sim \mathcal{N}(\mu, 1)$. This is the asymptotic MSE (scaled by the rate n) of the Hodge's estimator for different values of μ .

